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Considerations regarding superharmonic vibrations of a cracked beam and the variation in damping caused by the presence of the crack

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Abstract

A closing crack causes the dynamic behaviour of a vibrating system to be significantly nonlinear.The main distinctive features of such a vibrating system are the appearance of sub- and superharmonic resonances, and the significant nonlinearity of the vibration responses at sub- and superharmonic resonances (displacement, acceleration, strain, etc.). The nonlinear effects are much more sensitive to the presence of a crack than are either the change of natural frequencies and mode shapes or the generation of high harmonics in the spectrum of vibration at principal resonance or far from resonance. Thus, the appearance of sub- and superharmonic resonances may prove to be useful, highly sensitive indicators of a crack's presence at very early stages of its nucleation; moreover, the level of response nonlinearity in this regime may provide a quantitative evaluation of damage parameters (type, size and location).

At the same time, the manifestation of nonlinear effects depends not only on the crack parameters but also on the level of damping in a vibrating system. Recent experimental tests have revealed that crack nucleation and growth result in an increase of damping in a vibrating system. Consequently, the influence of crack's parameters upon the nonlinear effects should be determined while taking into account the change of damping in a vibrating system rather than assuming either constant damping or the total absence of damping.

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The empirical relationship between energy dissipated in a crack and nominal stress intensity factor range, estimated by curve-fitting experimental data, has been introduced into an FE model of a beam with a closing crack, and the influence of the damping level on the nonlinear dynamic behaviour of the beam was investigated.

1. Introduction

Increasingly over recent years, according to the literature available, attention has been focussed on the investigations of sub- and superharmonic resonance regimes of vibration for application to the problem of damage diagnostics [1–8], partly due to the fact that these regimes of vibration are highly sensitive to the presence of damage of fatigue crack type [1,4–8].

The mathematical modelling of such cracks is usually based on the assumption that a crack periodically closes and opens in the process of cyclic deformation of a structure (and, for this reason, it is often termed a ''closing'' or ''breathing'' crack), leading to the instantaneous change of structure stiffness. In this case, the change of the stiffness of the cracked structure is modelled by the unsymmetrical piecewise linear characteristic of the restoring force [9,10], or by specific modification of the driving force [11].

A closing crack causes the dynamic behaviour of a vibrating system to be significantly nonlinear, giving rise to a number of fundamental difficulties with regard to determining analytical solutions. Numerical analyses [1–6,8] and a small number of experimental investigations [1,6,8] on the forced vibrations of beams with a closing crack have demonstrated that the main distinctive features of such a vibration system are the appearance of sub- and superharmonic resonances and the significant nonlinearity of the vibration responses at sub- and superharmonic resonances (displacement, acceleration, strain etc.). Here, these phenomena are termed as 'the nonlinear effects'.

For instance, in Ref. [3], a step-sine test on cantilever beam with a closing crack was simulated and the amplitude response around the principal and superharmonic of order $2/1$ resonances was obtained (Fig. 1), the latter being the result of the essential nonlinearity of the restoring force characteristic. It is important to note that the amplitude of the superharmonic resonance is considerably less than the resonance amplitude.

The nonlinear effects are much more sensitive to the presence of a crack than either the change of natural frequencies and modeshapes (by one or even two orders of magnitude) or the generation of high harmonics in the spectrum of vibration at principal resonance or far from resonance [7,8].

Thus, the appearance of superharmonic resonances may prove to be useful as highly sensitive indicators of crack presence at very early stages of its nucleation; moreover, the level of vibration response nonlinearity in this regime may provide a quantitative evaluation of the damage parameters (type, size and location). In comparison with the even superharmonic resonance regimes, the subharmonic regimes of vibration are less efficient for diagnostics of small cracks [1,4,7] and therefore are not considered in the present paper.

However, as concerns the practical implementation of damage diagnostics using the superharmonic resonance regimes of vibration at least three essential obstacles must be negotiated.

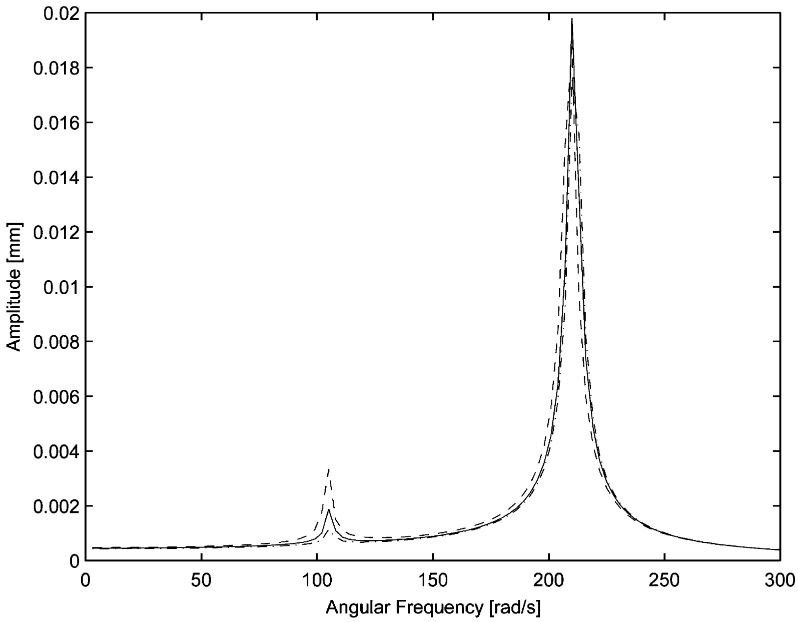

Fig. 1. Amplitude response of a cracked beam obtained by a step-sine test.

The first obstacle is associated with the fact that the mentioned regimes may arise in the presence of any type of nonlinearity of a vibrating system, including symmetrical elastic nonlinearity [12], geometrical nonlinearity [4], or as a result of nonlinear damping [13]. In subsequent sections of this article, it is assumed that for the cases considered, the influence of these nonlinear effects is negligible and do not cause the nonlinear regimes of vibration to be exhibited. It is considered by the authors that this assumption is true for most real structures at low levels of vibrations having internal damping, which is not dry friction in nature. Indeed the amplitudes of forced vibrations at a frequency far from the frequency of principal resonance are very small. In addition, as was shown in Ref. [4] using the example of forced vibrations of a cantilever beam, a threshold value of driving force exists below which it is not possible to excite the superharmonic vibrations of an undamaged structure in the presence of geometrical nonlinearity. Correspondingly, the unsymmetrical piecewise linear characteristic of the restoring force that models a fatigue crack and represents a particular case of elastic nonlinearity is related by the classification of Vulfson and Kolovsky [13] to the class of essentially nonlinear functions. This essential nonlinearity is exhibited even in the presence of very small cracks and can cause superharmonic vibrations [1,4,7] at very low amplitudes of driving force.

The second obstacle is particularly important from a practical point of view. As a matter of fact, it is difficult to ensure the rigorously harmonic excitation of vibrations. If the frequency of one of the harmonics in the spectrum of driving force coincides with the resonance frequency of a vibrating system, then the vibrations of a linear system similar to superharmonic resonance of order j/i will arise (the order j/i of nonlinear regime indicates how many natural periods of vibration j fall at i periods of external harmonic excitation [4]). The authors term this regime ‘pseudo-superharmonic’ because, in this case, both forced vibrations at the frequency of main driving harmonic and resonance vibrations excited by additional harmonics occur. Considering that in practice all excited systems cause distortion of a harmonic driving force to some extent, the need to prevent pseudo-superharmonic regime arises. The idea described in the work of Magone and Beresnevich [14] makes it possible to avoid the pseudo-superharmonic resonance vibration but, unfortunately, the work does not demonstrate the practical implementation of this concept.

In previously published experimental works [1,6] the possibility of exciting pseudo-superharmonic vibrations has not been considered and hence the extent of excitation system harmonicity has not been estimated. Consequently, this aspect, of fundamental importance for the practical feasibility of damage diagnostics based on the utilisation of superharmonic regime, is the subject of on-going research.

The essence of the third obstacle lies in the fact that, as has been shown in Ref. [7], the manifestation of nonlinear effects depends not only on the crack parameters but also on the level of damping in a vibrating system. The data of direct experimental investigations [15–19] attest that the fatigue crack growth is accompanied by a considerable increase in the damping characteristic of cracked specimens. Consequently, the influence of the crack parameters on the nonlinear effects should be determined by taking into account the change of damping in a vibrating system rather than assuming constant damping, which has been the case in the past [1–5], or without damping at all [9,11]. If the increase in damping is neglected, the prediction of damage magnitude will be erroneous.

Correspondingly, the aim of this work has been to investigate the influence of damping on the level of nonlinearity of the vibration response at the superharmonic resonance based on the finite element model of a beam with a closing crack which takes into consideration the change of damping due to the crack growth.

2. Model of cracked beam taking into account real energy dissipation in a crack

The mathematical model used for the cantilevered beam with a transverse one edge non-propagating closing crack is based on the Finite Element Model proposed in Ref. [2], and here, for the sake of completeness, it is presented briefly.

2.1. Stiffness matrix

According to the principle of Saint-Venant, the stress field is influenced only in the region adjacent to the crack. Consequently, the element stiffness matrix, with the exception of the terms representing the cracked element, may be regarded as unchanged under a certain limitation of the element size.

The general approach to the problem is that, for the cracked beam element, the elements situated on one side can be regarded as external forces (bending moment M and shear force P) applied to the cracked element, while the elements on the other side can be regarded as constraints. In this way the flexibility matrix can be easily calculated and then, from the conditions of equilibrium, the stiffness matrix of the cracked element can be derived. With shearing action neglected, the strain energy of an element of length l without a crack is

$$W^{(0)} = \frac{1}{2EI}\int_0^l (M + Px)^2 \, dx, \tag{1}$$

where E is Young's modulus and I is the inertia moment of the transversal section, while the additional stress energy due to the crack can be expressed as

$$W^{(1)} = b\int_0^a \{[(K_{IM} + K_{IP})^2 + K_{IIP}^2]/E'\} \, da \tag{2}$$

with

$$K_{IM} = (6M/bh^2)\sqrt{\pi a}F_I(s), \tag{3}$$

$$K_{IP} = (3Pl/bh^2)\sqrt{\pi a}F_I(s), \tag{4}$$

$$K_{IIP} = (P/bh)\sqrt{\pi a}F_{II}(s), \tag{5}$$

and

$$F_I(s) = \sqrt{(2/\pi s)\tan(\pi s/2)}[0.923 + 0.199(1 - \sin(\pi s/2)^4)]/\cos(\pi s/2), \tag{6}$$

$$F_{II}(s) = (3s - 2s^2)(1.122 - 0.561s + 0.085s^2 + 0.18s^3)/\sqrt{1 - s}, \tag{7}$$

where $E' = E$ for plane stress, $E' = E/(1 + v)$ for plane strain, v is the Poisson ratio, $s = a/h$ is the relative crack size, a is the crack depth, h and b are the height and the width of the cross-section, respectively and K_I and K_{II} are stress intensity factors for opening type and sliding type cracks, respectively.

The term $c_{ik}^{(0)}$ of the flexibility matrix $\mathbf{C}_e^{(0)}$ for an element without crack is derived by means of Castigliano's theorem in the linear elastic range:

$$c_{ik}^{(0)} = \frac{\partial^2 W^{(0)}}{\partial P_i \partial P_K} = c_{ki}^{(0)}, \quad i, k = 1, 2 \quad P_1 = P, \; P_2 = M. \tag{8}$$

The term $c_{ik}^{(1)}$ of the additional flexibility matrix $\mathbf{C}_e^{(1)}$ due to the crack is calculated in the same way:

$$c_{ik}^{(1)} = \frac{\partial^2 W^{(1)}}{\partial P_i \partial P_k} = c_{ki}^{(1)}, \quad i, k = 1, 2 \quad P_1 = P, \; P_2 = M. \tag{9}$$

Consequently, the term c_{ik} of the total flexibility matrix for the damaged element $\mathbf{C}_e$ is

$$c_{ik} = c_{ik}^{(0)} + c_{ik}^{(1)}. \tag{10}$$

From the equilibrium condition

$$(P_i \;\; \mathrm{M}_i \;\; \mathrm{P}_{i+1} \;\; \mathrm{M}_{i+1})^{\mathrm{T}} = \mathbf{T}(P_{i+1} \;\; M_{i+1})^{\mathrm{T}}, \tag{11}$$

where

$$\mathbf{T} = \begin{bmatrix} -1 & 0 \\ -l & -1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}. \tag{12}$$

By the Principle of the Virtual Work, the stiffness matrix of the undamaged element is

$$\mathbf{K}_e = \mathbf{T}\mathbf{C}_e^{(0)-1}\mathbf{T}^{\mathrm{T}}, \tag{13}$$

or

$$\mathbf{K}_e = \frac{EI}{l^3}\begin{bmatrix} 12 & 6l & -12 & 6l \\ 6l & 4l^2 & -6l & 2l^2 \\ -12 & -6l & 12 & -6l \\ 6l & 2l^2 & -6l & 4l^2 \end{bmatrix}, \tag{14}$$

while the stiffness matrix of the cracked element may be derived as

$$\mathbf{K}_e = \mathbf{T}\mathbf{C}_e^{-1}\mathbf{T}^{\mathrm{T}}. \tag{15}$$

The global stiffness matrix $\mathbf{K}$ is obtained assembling the elements stiffness matrices $\mathbf{K}_e$.

2.2. Mass matrix

In order to evaluate the dynamic response of the cracked beam when acted upon by an applied force, it is supposed that the crack does not affect the mass matrix $\mathbf{M}$. Therefore, for a single element:

$$M_e = \frac{ml}{420}\begin{bmatrix} 156 & 22l & 54 & -13l \\ 22l & 4l^2 & 13l & -3l^2 \\ 54 & 13l & 156 & -22l \\ -13l & -3l^2 & -22l & 4l^2 \end{bmatrix}, \tag{16}$$

where m is the mass per unity length of the beam.

2.3. Damping matrix

For the purpose of this article, particular attention must be paid to the calculation of the damping matrix $\mathbf{D}$.

Considering a *proportional damping* model [20], the damping matrix $\mathbf{D}$ has been calculated through the inversion of the mode shape matrix relative to the undamaged structure:

$$\mathbf{D} = (\mathbf{\Phi}^{\mathrm{T}})^{-1}\mathbf{d}\mathbf{\Phi}^{-1}, \tag{17}$$

where

$$\mathbf{d} = 2\begin{bmatrix} \zeta_1\omega_1 M_1 & 0 & \dots & 0 \\ \dots & \zeta_2\omega_2 M_2 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \zeta_n\omega_n M_n \end{bmatrix}, \tag{18}$$

in which ζ_i is the ith modal damping ratio, ω_i is the ith natural frequency and M_i is the ith modal mass relative to the undamaged beam.

The modal damping ratio of the cracked beam is related to the *logarithmic decrement of vibrations* δ_c (LDV) used in the experimental tests (see Section 3) by the following expression:

$$\zeta_i = \delta_c/2\pi. \tag{19}$$

Taking into account the relationship between the LDV of the undamaged beam (initial level of damping) δ and the damping ratio ψ,

$$\delta(\sigma_a) \cong \frac{1}{2}\psi(\sigma_a),$$

where $\psi(\sigma_a) = \Delta U(\sigma_a)/U(\sigma_a)$, it is possible to express δ_c of the specimen with a closing crack in the following way [20]:

$$\delta_c(\sigma_a) \cong \frac{\Delta U(\sigma_a) + \Delta U_c(\Delta K_{\mathrm{I}})}{2U(\sigma_a)} = \delta(\sigma_a) + \frac{\Delta U_c(\Delta K_{\mathrm{I}})}{2U(\sigma_a)}. \tag{20}$$

In formula (20) δ is the LDV of the undamaged beam (initial level of damping), σ_a is the maximum stress amplitude, ΔU_c is the energy dissipated in the crack per cycle, U is the strain energy of the beam and ΔK_{I} is the nominal stress intensity factor range. The strain energy for the cantilever beam without the end mass was calculated by

$$U(\sigma_a) = \frac{bhL}{18E}\sigma_a^2, \tag{21}$$

where L is the length of the beam.

As a result of experimental investigation [21] it was shown that the absolute value of energy dissipation in a non-propagating fatigue crack is uniquely determined by the variation of the nominal stress intensity factor range ΔK_{I} by the following equation:

$$\Delta U_c = b \cdot (8.634675 \times 10^{-5}\Delta K_{\mathrm{I}} + 3.87315 \times 10^{-4}\Delta K_{\mathrm{I}}^2 - 1.29826 \times 10^{-5}\Delta K_{\mathrm{I}}^3). \tag{22}$$

The value of ΔK_{I} for the case of symmetric vibrations was determined by the formula [22]

$$\Delta K_{\mathrm{I}} = \sigma_a^c\sqrt{\pi a}(1.122 - 1.40s + 7.33s^2 - 13.08s^3 + 14.0s^4), \tag{23}$$

where σ_a^c is the nominal stress amplitude in the cracked cross-section.

2.4. Equation of motion

When the crack closes and its interfaces are completely in contact with each other, the dynamic response can be determined directly as that of the uncracked beam. However, when the crack

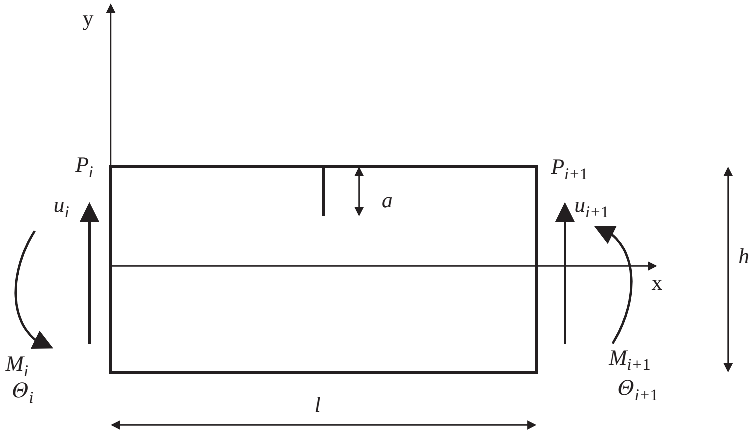

Fig. 2. The cracked element.

opens the stiffness matrix of the cracked element should be introduced in replacement at the appropriate rows and columns of the general stiffness matrix.

Under the action of the excitation force **F**, crack opening and closing alternate in time, making the equations of motion of the cracked beam nonlinear:

$$\mathbf{M}\ddot{\mathbf{u}} + \mathbf{D}\dot{\mathbf{u}} + \mathbf{K}\mathbf{u} = \mathbf{F}, \tag{24}$$

where

$$\mathbf{K} = \mathbf{K_u} - \gamma\Delta\mathbf{K} \tag{25}$$

with

$$\Delta\mathbf{K} = \mathbf{K_u} - \mathbf{K_d}, \tag{26}$$

denoting the changes in the global stiffness matrix due to the crack and

$$\gamma = \begin{cases} 1 & \text{when the crack is open,} \\ 0 & \text{when the crack is closed.} \end{cases} \tag{27}$$

Since an exact solution of these equations does not exist, a numerical method must be adopted to simulate the dynamic behaviour of the cracked beam, proceeding step-by-step in time. In such a simulation, to determine the state of the crack, i.e. whether opened or closed, it is sufficient to evaluate the slopes θ_i, θ_{i+1}, of the response deformation at the 'control' nodes, $i, i+1$, closest to the crack (where i is the node closer to the fixed end of the beam, as shown in Fig. 2). For a crack on the upper side of the beam, the condition of crack closing is then equivalent to $\theta_i < \theta_{i+1}$.

According to Bathe and Gracewski [23], it is possible to write an incremental form of the nonlinear equation of motion for the cracked beam that can be solved with an implicit time integration scheme and the modified Newton iteration.

3. Validation of the model by the results of tests

The results of four specimens of materials, possessing different damping properties, were used to determine the validity of the FE model of a beam with a closing crack presented in Section 2.

The tests were performed using the experimental set-up KD-1M [24]. The set-up KD-1M was devised for the determination of the damping characteristic of specimens with high accuracy and resolving ability, for the high-cycle fatigue testing of specimens and for the execution of spectral analysis of the vibration response of strain and acceleration at the principal and superharmonic vibration resonances of the specimens.

The vibrating system consists of the specimen rigidly fixed in the massive (approx. half-ton) frame, which is suspended on thin steel wires in order to isolate the vibrating system (Fig. 3). The resonance method was used to excite the first bending mode of vibration by means of an electromagnetic system including a waveform generator and power amplifier. The specimens were tested in the plane of maximal resistance to bending, with ferromagnetic plates being attached on the end of the specimens to create the interaction with the electromagnets. The measuring system consisted of a strain gauge and accelerometer, the signals from which were passed through the amplifiers, visualised using a double-channel oscilloscope, and acquired for subsequent data-processing using a computer-based system.

Fatigue cracks were grown from sharp concentrators through fatigue testing. The measurements of the depths of cracks were executed by an optical microscope, the absolute error being ± 0.1 mm. The strain gauges were polished before testing for the better observation of a crack.

The dimensions of the specimens gauge length and mechanical properties of the materials are shown in Table 1, where L_c, L_{ac} and L_{sg} are the locations of the crack, the accelerometer and the strain gauge; ρ is the density; σ_{-I} and σ_y are the fatigue limit and the yield stress of materials, respectively.

At stress levels, which are of interest for practical engineering applications, the damping characteristics of materials and structural elements are dependent on the stress amplitude. Therefore, the investigations of damping properties of cracked specimens were conducted based on the analysis of amplitude dependencies of damping characteristics. These dependencies were determined by the free oscillation method and, as regards the damping characteristic, the LDV was used:

$$\delta = \frac{1}{N} \ln\left(\frac{\alpha_i}{\alpha_{i+N}}\right), \tag{28}$$

where α_i and α_{i+N} are the amplitudes of ith and $(i+N)$th cycles of vibration, respectively. The relative error of the LDV determination did not exceed 10%.

In general, commercially available spectrum analysers offer insufficient resolution to permit the evaluation of the presence of a closing crack through nonlinear distortions of the vibration response at resonance regimes of vibration. For this reason, a PC-based highly sensitive system for spectrum analysis of different vibration responses and special software were developed [24]. This system was also used for the spectrum analysis of vibration responses at superharmonic resonance and for the determination of the LDV.

It is common practice to investigate the forced vibrations of different systems based on the assumption that a driving force is harmonic. However, it is extremely difficult in practice to

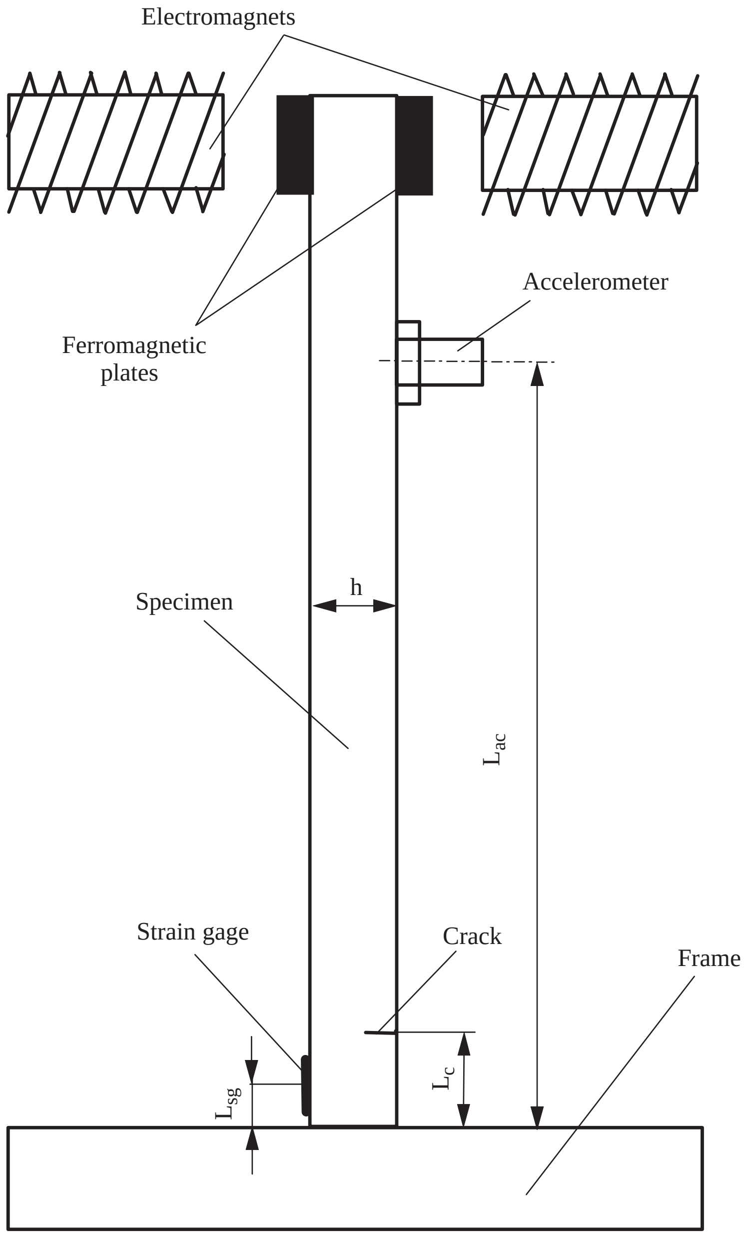

Fig. 3. Schematic diagram of the experimental set-up KD-1M.

generate religiously harmonic excitation of vibrations. In reality, the spectrum of driving force contains higher harmonics. As a consequence, if the frequency of the driving force is several times lower than the frequency of principal resonance, it may give rise to the so-called pseudo-superharmonic vibrations, which are similar in appearance to the superharmonic ones. Such a regime of vibrations takes place when the frequency of one of the high harmonics in the spectrum

Table 1
Dimensions of specimens and mechanical properties of materials

Material of a specimen	L	L_c	L_{ac}	L_{sg}	h	b	E	ρ	ν	σ_{-1}	σ_y
#	mm	mm	mm	mm	mm	mm	Gpa	kg/m^3	#	MPa	MPa
Titanium alloy VT-8	230	13	142	9	20	4	127	4480	0.3	500	950
Carbon steel 3	230	14	138	8	20	4	200	7800	0.26	140	N/A
Duralumin alloy D-16	230	14	142	7	20	4	71	2800	0.31	130	290
Cu-Al alloy	230	2	142	7	20	4	116	7500	0.29	185	350

of driving force coincides with the frequency of principal resonance. It should be noticed that in contrast to the pseudo-superharmonic vibrations, the excitation of pseudo-subharmonic vibrations is impossible in principle. From this point of view, the subharmonic regimes of vibration are more reliable than superharmonic ones as applied for damage detection but less sensitive to the presence of small cracks.

Since the investigations concentrated mainly on the superharmonic resonance of order 2/1, it was necessary to avoid the presence of second harmonic in the spectrum of driving force. The practical realisation of this idea depends on the way in which the dynamic load is applied. In our case, that is based on the use of the electromagnetic system of excitation, a solution was found by using two electromagnets disposed on opposite sides of the specimen; the voltage was applied to them with a respective phase shift of $\pi/2$ and, as a consequence, the resulting excitation force can be expressed by the following formula:

$$F = q_0\left\{\left|\sin\frac{pt}{2}\right| - \left|\sin\left(\frac{pt-\pi}{2}\right)\right|\right\}. \tag{29}$$

It is shown in Fig. 4 by the solid line. In Eq. (29) the modulus of functions reflects the fact that each electromagnet can produce only attractive force. The expansion of this function into the Fourier series

$$\begin{aligned} F &= -\frac{8}{\pi}q_0\left(\frac{1}{3}\cos pt + \frac{1}{35}\cos 3pt + \frac{1}{99}\cos 5pt + \cdots\right) \\ &\approx -q_0(0.848\cos pt + 0.073\cos 3pt + 0.026\cos 5pt + \cdots) \end{aligned} \tag{30}$$

confirms that function (29) does not include the even harmonics. In such a way in tests of specimens the appearance of pseudo-superharmonic resonance of order 2/1 was avoided.

Fig. 5 demonstrates the feasibility of predicting the changes of the LDV of specimens caused by crack growth. The initial amplitude dependencies of the LDV of specimens (that is in an undamaged state) are shown by the dashed line. These curves were used for the calculation of the damping characteristic of the cracked specimens with the use of formula (20). As can be seen from Fig. 5, the damping properties of a cracked beam can be predicted satisfactorily in this way. Furthermore, the results of experiments and calculations corroborate the conclusion made in Ref. [25] that the lower the initial level of damping of the structure, the more the damping increases

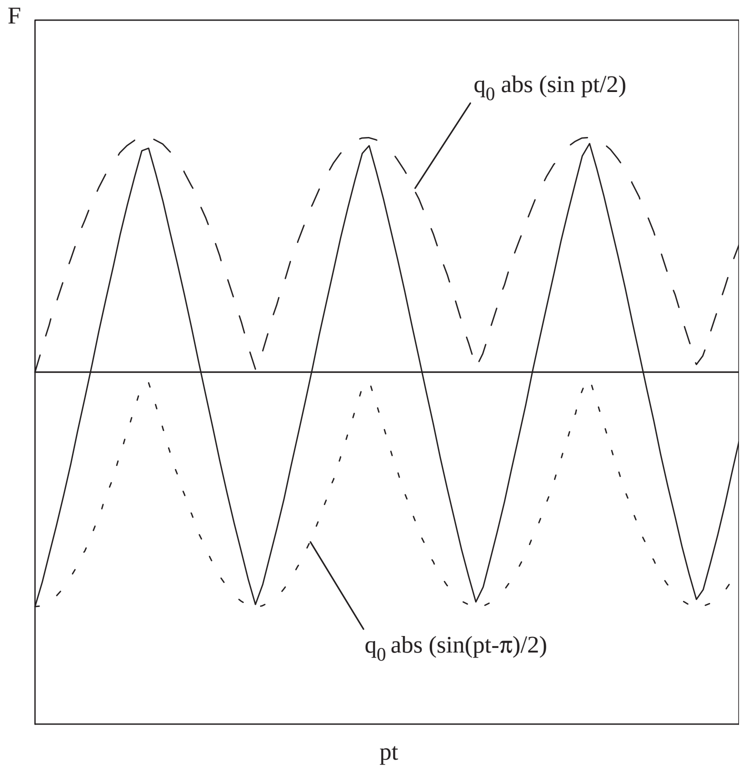

Fig. 4. Forces produced by the first (dash line) and the second (dot line) electromagnets and resulting driving force of two electromagnets (solid line).

following crack initiation. Of the four materials characterised, titanium alloy VT-8 possesses the lowest damping properties; thus, it is quite logical that the increase in the LDV for this specimen LDV with crack growth was the most significant in this case—over three times (Fig. 5(a)).

Table 2 compares experimental data with the results obtained using an FEM with damping determined from Eq. (20) (the term σ_a in Table 2 is the stress amplitude of forced vibrations in experimental investigations of nonlinear effects).

It can be observed that the change of natural frequencies of cracked specimens can be predicted with accuracy and the difference between the results of experiments and calculations regarding the LDV of specimens is less than 20%; furthermore, the discrepancy in the second harmonic of the acceleration response A_2 at the principal and superharmonic resonances was less than 35% and 30%, respectively. Moreover, it can be noted that almost all the results of second harmonic calculations at both resonances exceed the experimental data, which can be explained by the slight difference between the frequency of the driving force and the super-resonance frequency of the vibrating system in the tests. In this case, the extent of nonlinearity of the vibration response is of a somewhat lower level [7].

Thus, it can be concluded that the FE model of the beam with a closing crack proposed here can satisfactorily predict the damping properties and level of nonlinearity of its forced vibrations,

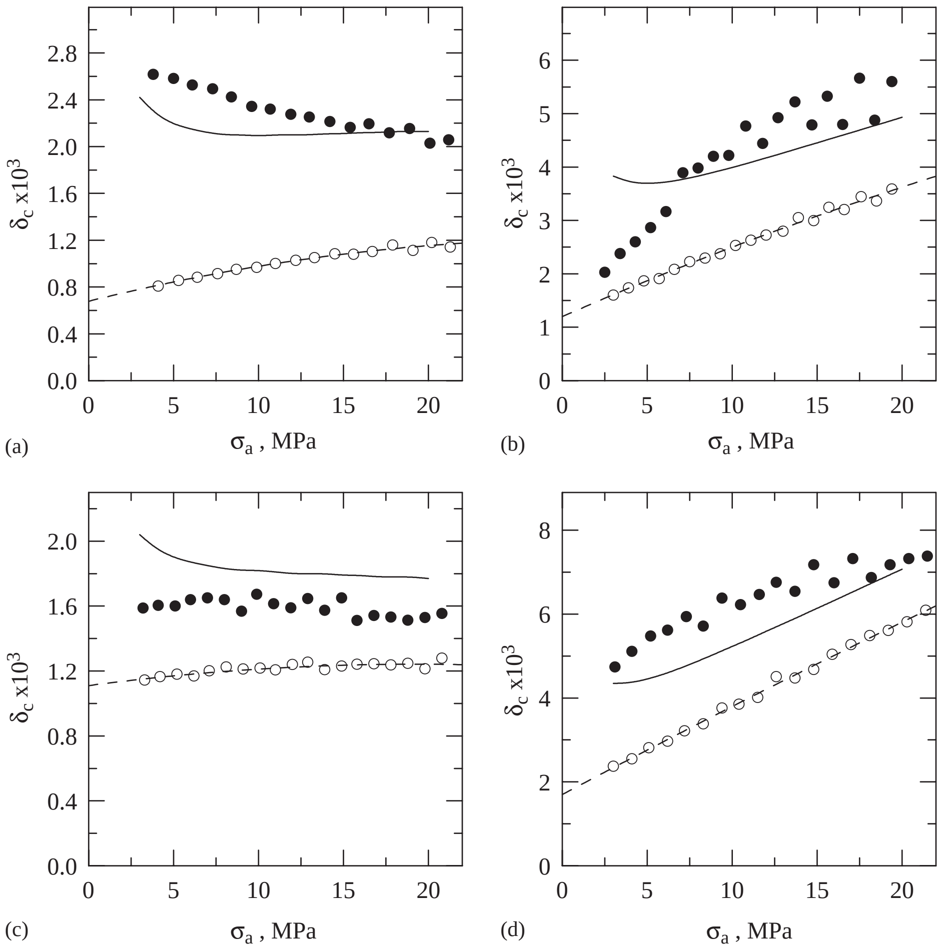

Fig. 5. Amplitude dependency of the LDV of titanium alloy VT-8 specimen (a), carbon steel 3 specimen (b), duralumin alloy D-16 specimen (c), Cu–Al alloy specimen (d): ○○○ , intact specimens (experiment); ●●●, cracked specimens (experiment); –––, intact specimens (the results of approximation); ——, cracked specimens (the results of calculations).

Table 2
Validation of the model by the results of the tests

Material of a specimen	a/h	L_c/L	ω_c/ω		σ_a (MPa)	δ_c		A_2/A_1 (acceleration)			
			Experiment	Theory		Experiment	Theory	Resonance		Super-resonance 2/1	
								Experim.	Theory	Experim.	Theory
VT-8	0.10	0.056	0.994	0.994	9.0	0.0024	0.0021	0.011	0.012	5.16	4.68
St.3	0.14	0.061	0.994	0.994	7.3	0.0035	0.0038	0.008	0.009	4.88	6.47
D-16	0.15	0.061	0.995	0.995	14.1	0.0016	0.0018	0.008	0.011	6.29	8.99
Cu-Al	0.21	0.009	0.981	0.981	8.7	0.0060	0.0050	0.015	0.023	6.97	9.08

and can therefore be used for the investigations of nonlinear effects in conditions of variations in the damping level in a vibrating system.

4. Nonlinear effects at superharmonic vibrations

4.1. The influence of damping on the nonlinear effects at principal and superharmonic resonances

The influence of damping on the characteristics of nonlinearity of the beam vibrations was investigated using the somewhat simplified model of the beam, in which damping was assumed to be independent of the amplitude of vibration and crack size. The idea was to investigate the influence of one parameter only (the level of LDV) on the following nonlinear characteristics relative to the vibrational displacement response at the end of the beam $u(t)$:

$$R_1 = a_0/A_1 \text{ —zero coefficient ratio;}$$

$$R_2 = A_2/A_1 \text{ —second harmonic coefficient ratio;}$$

$$R_3 = \sum_{k=2}^{10} A_k/A_1 \text{ —harmonics coefficient ratio,}$$

where the coefficient a_0 and amplitudes of harmonics A_k are taken from the Fourier series

$$u(t) = \frac{a_0}{2} + \sum_{k=1}^{\infty} A_k \sin(k\omega_F t + \varphi_k),$$

in which ω_F is the circular frequency of the exciting force.

For comparison of the sensitivity of different damage indicators also the ratio $R_4 = \omega_c/\omega$ has been determined, where ω_c is the natural circular frequency of the structure with a closing crack:

$$\omega_c = \frac{2\omega\omega_o}{\omega + \omega_o}.$$

In the above formula, ω and ω_o are the natural circular frequencies of the intact beam and of the beam with an open crack, respectively.

For comparison of the sensitivity of different damage indicators the ratio $R_4 = \omega_o/\omega$ has also been determined.

The calculations were performed at three levels of the LDV ($\delta = 0.0005$, 0.005 and 0.05) at resonance and super-resonance first mode bending vibrations of the beam. The geometrical and mechanical properties of the beam are $L = 0.2\,\text{m}$, $L_c/L = 0.1$, $h = 0.02\,\text{m}$, $b = 0.004\,\text{m}$, $E = 2.06 \times 10^{11}\,\text{N/m}^2$ and $\rho = 7850\,\text{kg/m}^3$.

As can be seen from Fig. 6, the relative values of second harmonic R_2 and zero coefficient R_1 at principal resonance are very small; therefore, to enable reliable evaluation, particularly at small sizes of crack, it is necessary to use a highly sensitive system for spectral analysis. At the same time, at superharmonic resonance, even in the case of the smallest crack depth under investigation ($a/h = 0.1$), the nonlinear distortion of the beam vibration response is so large that it may be

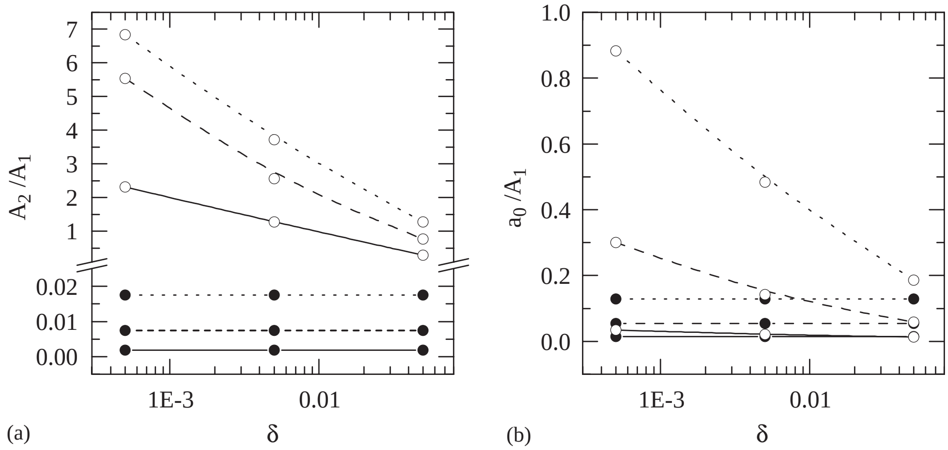

Fig. 6. Damping level dependency of the second harmonic (a) and zero coefficient (b) at principal (solid symbols) and superharmonic (open symbols) resonances: ——, $a/h = 0.1$; – – –, $a/h = 0.2$; ·········, $a/h = 0.3$.

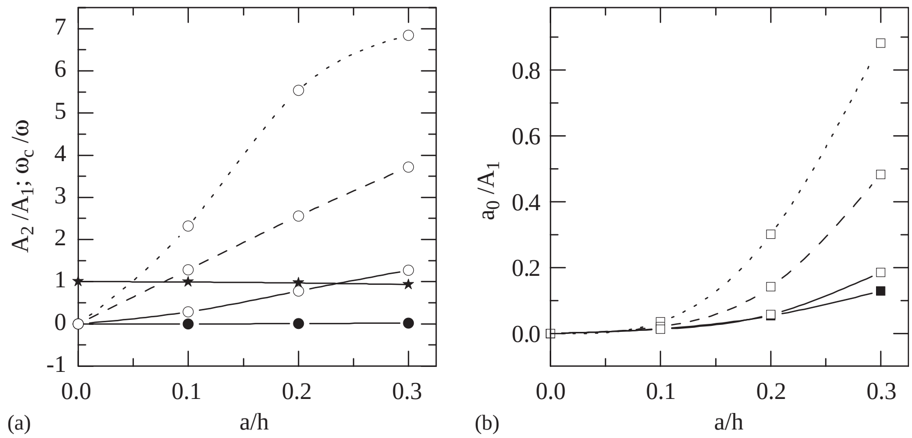

Fig. 7. Crack depth dependency of the second harmonic and natural frequency (a) and zero coefficient (b) at principal (solid symbols) and superharmonic (open symbols) resonances: ——, $\delta = 0.05$; – – –, $\delta = 0.005$; ·········, $\delta = 0.0005$; ★——★——★, ω_c/ω.

easily identified directly even from the screen of oscilloscope since, as at the minimal level of damping being considered ($\delta = 0.0005$), the amplitude of the second harmonic is already 2.3 times the amplitude of the main resonance (Fig. 7).

At the principal resonance, the level of damping practically does not influence the nonlinear effects. At the same time, at superharmonic resonance, the growth of damping essentially suppresses the dependence of all characteristics of nonlinearity on the relative crack size. The second harmonic decreases more intensively when the crack size is smaller.

For instance, at relative crack depth $a/h = 0.1$, the variation of the LDV from $\delta = 0.0005$ to $\delta = 0.05$ results in a decrease in the ratio R_2 by a factor of 8.0 and at $a/h = 0.3$ by a factor of 5.4. In contrast, the ratio R_1 decreases up to 2.7 times at $a/h = 0.1$ and up to 4.8 times at $a/h = 0.3$. In this case the harmonics coefficient R_3 exceeds the relative value of the second harmonic coefficient R_2 up to 26% at principal and up to 6% at superharmonic resonances of the beam.

It is important to note that there is no linear relation between the extent of damping growth and the magnitude of the variation in the nonlinear coefficients. For instance, an increase in the LDV of two orders of magnitude results in the decrease in nonlinear coefficients values by a factor of approximately 8.

As can be noted from Fig. 7, the crack growth in the range being investigated causes the essential increase in the second harmonic and zero coefficient in the spectrum of the displacement response of the beam at superharmonic resonance. At principal resonance the change of nonlinear characteristics as well as the change of natural frequency of the beam are considerably smaller being hardly noticeable against the scale used in Fig.7. It is also evident that the second harmonic in the spectrum of the displacement response at superharmonic resonance is most sensitive to the presence of the crack. To verify this assertion, the quantitative estimation of the extent of variation of all above-mentioned characteristics was determined using the intensity measure of respective functions at unitary variation of crack size:

$$V(s) = \frac{\partial R(s)}{\partial s}, \tag{31}$$

where R is the function expressing the dependence of nonlinear characteristics or natural frequency on the crack size.

The most significant results of the comparative analysis functions are shown in Fig. 8. As might be expected, the change in the crack depth function of the second harmonic at superharmonic resonance is the most sensitive. The intensity of change in this function exceeds by up to three orders of magnitude (depending on the level of damping) the intensity of change of the second harmonic at resonance and of the natural frequency of the beam. The zero coefficient at superharmonic resonance has sensitivity comparable to the second harmonic as regards the crack presence only at minimal damping, and practically coincides with the function for the second harmonic at the maximum level of damping being investigated.

It must be emphasised that in a number of cases an essential quantitative and qualitative change of intensity of nonlinear coefficients as far as crack growth may take place. For instance, at minimal damping the intensity of the second harmonic change from the beginning sharply increases and then drops almost to zero level, that is at $a/h > 0.3$ the change of the second harmonic is practically insignificant. At the same time, at middle and maximal damping the sensitivity of the second harmonic to the crack presence, being still high for the aims of damage diagnostics, depends faintly on the crack size.

Thus, the intensity of change of the characteristics of nonlinearity at superharmonic resonance is much higher than at the principal one. At superharmonic resonance the most changeable characteristics with respect to crack growth characteristics are the second harmonic R_2 and coefficient of harmonics R_3 and at principal resonance—zero coefficient R_1. It follows from Table 3 that the sensitivity of R_2 of the end of the beam displacement response at minimal level of damping exceeds 1218.5 times that at resonance (in the table subscripts "s" and "r" signify the superharmonic and principal resonant regimes of vibration, respectively). The difference of

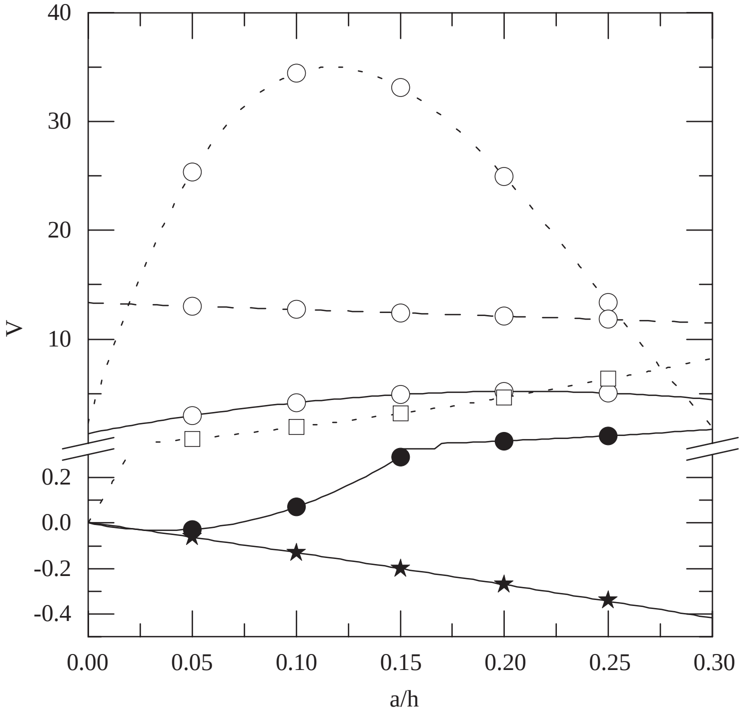

Fig. 8. The effect of crack depth on the velocity of functions $R(\gamma)$ change: —○—○—, R_2 (superharmonic resonance, $\delta = 0.05$); –○–○–, R_2 (superharmonic resonance, $\delta = 0.005$); ······○······○······, R_2 (superharmonic resonance, $\delta = 0.0005$); ················, R_1 (superharmonic resonance, $\delta = 0.0005$); —●—●—, R_2 (principal resonance); ★——★——★, R_4.

Table 3
The relative intensity of characteristics of nonlinearity change at superharmonic and principal resonances

δ	a/h	$(R_1)_s/(R_1)_r$	$(R_2)_s/(R_2)_r$	$(R_3)_s/(R_3)_r$
0.0005	0.1	2.5	1218.5	969.4
	0.2	5.6	748.6	609.9
	0.3	6.9	391.0	318.7
0.005	0.1	1.4	673.2	536.0
	0.2	2.6	345.6	283.0
	0.3	3.8	212.2	172.6
0.05	0.1	0.9	144.8	116.7
	0.2	1.1	104.5	87.2
	0.3	1.4	72.7	61.0

sensitivity drops almost to two orders of magnitude at increase of damping and crack depth. The zero coefficient R_1 at superharmonic resonance exceeds its sensitivity at resonance (up to 7 times) only at minimal damping.

4.2. Nonlinear effects taking into account the real energy dissipation in a crack

Essentially, the appearance of a crack in an elastic body results in an increase in its damping characteristic [7]. The model presented here of the cracked beam takes this into account and, as was shown in Section 3, predicts with sufficient accuracy the change of damping properties of the beam due to crack growth—the so-called real damping.

Assuming that the damping for the undamaged beam is initially of low level ($\delta = 0.0005$) and independent of the amplitude of vibration, the dependency of the LDV on the stress amplitude obtained with the model of the damaged beam for different crack depths is shown in Fig. 9. As can be seen, at such a low initial level of damping, the presence of a relatively small crack results in a significant increase in the LDV of the beam: approximately 12.8 times at $\sigma_a = 2\,\mathrm{MPa}$, 8.2 times at $\delta_a = 10\,\mathrm{MPa}$ and 7.4 times at $\sigma_a = 20\,\mathrm{MPa}$. In these conditions, the crack growth modifies the amplitude dependencies of the LDV in an unusual way: the LDV of the cracked beam increases sharply at small stresses (the explanation of this phenomenon was given in Ref. [25]). This phenomenon means that the most essential change of damping characteristic due to the crack growth takes place at small stress amplitudes.

It follows from Fig. 10 that the initial level of the LDV of the beam determines the extent of change of its damping capacity with crack growth. The higher the initial level of damping, the lower the change of damping characteristic of the beam with crack growth, assuming all other

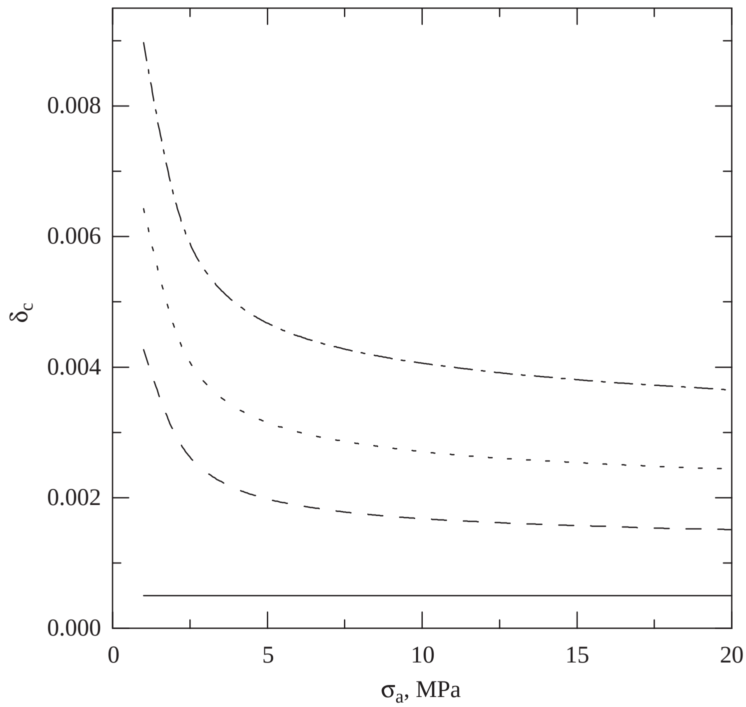

Fig. 9. The amplitude dependency of the LDV of the beam at different crack depths: ———, $a/h = 0$; – – – –, $a/h = 0.1$; · · · · · · · ·, $a/h = 0.2$; –·–·–·–, $a/h = 0.3$.

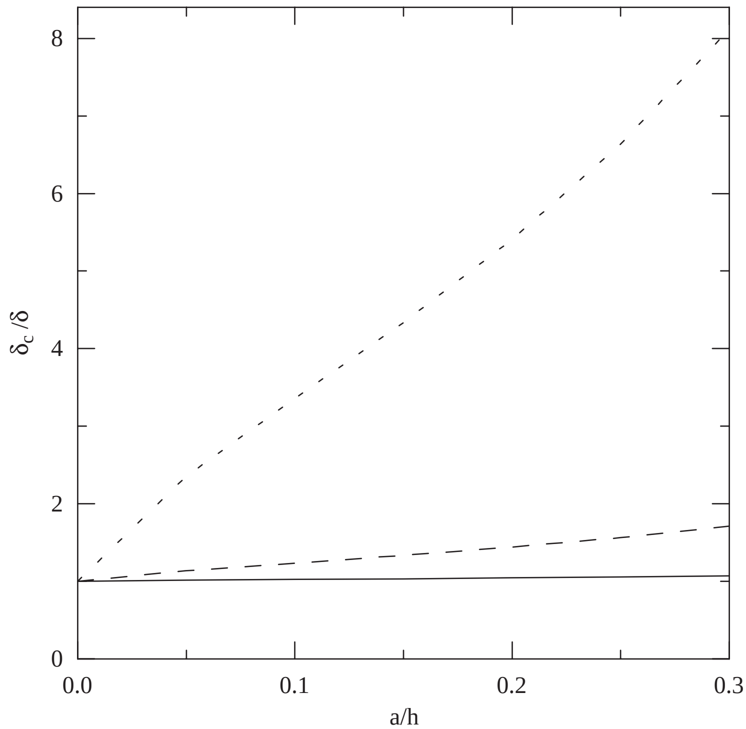

Fig. 10. The crack depth dependency of the LDV of the beam at different initial levels of damping: ———, $\delta = 0.05$; – – – –, $\delta = 0.005$; · · · · · · · ·, $\delta = 0.0005$.

Table 4
Characteristics of nonlinearity of displacement response at the free end of the beam at real damping ($\delta = 0.0005$; $a/h = 0.3$)

σ_a (MPa)	δ_c	Principal resonance			Superharmonic resonance 2/1		
		R_1	R_2	R_3	R_1	R_2	R_3
5	0.0047	0.1284	0.0175	0.0221	0.5143	3.9583	4.0976
10	0.0041	0.1284	0.0175	0.0221	0.5331	4.1034	4.2454
20	0.0037	0.1284	0.0175	0.0221	0.5625	4.3392	4.4866

factors remain the same. At sufficiently high levels of damping even a relatively deep crack does not exert a noticeable influence on the damping characteristic of the beam.

The beam model, accounting for energy dissipation in the crack, was used for the investigation of nonlinear distortions of vibration of the end of the beam with initial damping $\delta = 0.0005$ and crack depth $a/h = 0.3$, at three levels of stress amplitudes: $\sigma_a = 5$, 10 and 20 MPa. Table 4 illustrates the results of calculations at principal and superharmonic resonances.

As can be seen, the values of the coefficients R_1, R_2 and R_3 at principal resonance do not depend on the stress amplitude and damping level; therefore, they coincide exactly with the corresponding values of the nonlinear characteristics in Fig. 6. At superharmonic resonance, the nonlinearity of the vibration response is significantly greater. For instance, the zero coefficient R_1 exceeds its values at the principal resonance by a factor between 4.0 and 4.4, the amplitude of second harmonic R_2—by a factor of 226–248 and the harmonics coefficient R_3—by a factor of 185.4–203.0. At the same time all characteristics of nonlinearity at the superharmonic resonance are dependent on the stress amplitude since, in this case, the level of stress determines the level of the damping in a system. It follows from the results shown in Table 4 that the increase in stress amplitude causes the decrease in the LDV of the beam—by approximately 27% over the range of stress amplitudes being considered. Such a small decrease in damping results in a smaller change of the characteristics of nonlinearity—approx. 10%. Clearly, from a practical point of view, the inter-relationship between the characteristics of nonlinearity and the stress amplitude is not an essential aspect in the range of stresses being considered.

A fundamental conclusion follows from these results: for the same crack size, but at different stress amplitudes, various manifestations of nonlinear effects may take place. It is also necessary to bear in mind that at small stress amplitudes, as follows from Fig. 9, a sharp increase in the LDV takes place, which may cause a significant change of the characteristics of nonlinearity. Neglecting this phenomenon will result in a fundamental error in the estimation of damage size. The solution of the inverse problem of damage diagnostics, based on the measured characteristics of nonlinearity, if the change of damping caused by a crack growth is not taken into account, will result in an underestimation of the predicted value of damage size.

It is necessary to underline that the presented model of the beam can be used for the solution of direct and inverse problems of damage diagnostics, in conditions of a non-propagating crack. Otherwise the application of the theory may result in substantial and unpredictable errors.

The use of acceleration response can considerably increase the sensitivity of characteristics of nonlinearity to the presence of a crack by up to 4 times with regard to the displacement response, corroborated by the results of experiments and calculations presented in Table 2.

5. Conclusions

A mathematical model of the beam with a closing crack was developed which takes into account the energy dissipated in a crack by means of the relationship between this energy and the nominal stress intensity factor range. The model makes it possible to predict the changes of the damping characteristic of the beam caused by the crack presence and at this condition calculates its nonlinear behaviour.

A mathematical model of a beam with a closing crack was developed, which takes into account the relationship between the energy dissipated in a crack and the nominal stress intensity factor range. The model makes it possible to predict the changes of damping of cracked beam caused by the crack presence and at this condition calculates its nonlinear behaviour.

It was shown both numerically and experimentally that the characteristics of nonlinear distortion of vibrations at superharmonic resonance of the beam of order 2/1 (especially second harmonic and coefficient of harmonics) are very sensitive to the presence of closing crack. At the same time they are strongly dependent not only on the crack parameters but also on the level of damping in a vibrating system. The higher the level of damping, the lower the manifestation of nonlinear effects.

The appearance of nonlinear effects when the structure is harmonically excited at a frequency, which is a submultiple of a natural frequency, can be used to detect the presence of very small closing cracks. However, the extent and position of the crack can be determined by using these nonlinear behaviour only if the real level of damping in a cracked structure is accounted for.

As it seems to the authors of the present paper, one of the most complex problems in the way of practical application of nonlinear effects for the damage diagnostics is the avoidance of pseudo-superharmonic resonances. One possible solution to the problem was shown in the present paper. But it is necessary to bear in mind that the adjustment of this excitation system to satisfy Eq. (29) is extremely laborious and can hardly be recommended for wide application. Therefore, the authors intend to continue the development of methods that will make it possible to prevent the excitation of pseudo-nonlinear resonances.

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